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Advanced Statistical Energy Analysis

K. H. Heron

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Advanced statistical energy analysis

By K. H. Heron
Defence Research Agency, Farnborough, Hampshire GU14 6TD, U.K.

A high-frequency theory (advanced statistical energy analysis (ASEA)) is developed which takes account of the mechanism of tunnelling and uses a ray theory approach to track the power flowing around a plate or a beam network and then uses statistical energy analysis (SEA) to take care of any residual power. ASEA divides the energy of each sub-system into energy that is freely available for transfer to other sub-systems and energy that is fixed within the sub-system. The theory allows for coupling between sub-systems that are physically separate and can be interpreted as a series of mathematical models, the first of which is identical to standard SEA and subsequent higher order models are convergent on an accurate prediction. Using a structural assembly of six rods as an example, ASEA is shown to converge onto the exact results, whereas SEA is shown to overpredict by up to 60 dB.

1. Introduction

Statistical energy analysis (SEA) has been successfully applied to many noise and vibration problems. In particular SEA has become very useful as a framework for interpreting a vibro-acoustic data base. SEA often leads to a better understanding of the problem and SEA can point the way to practical solutions. However, when used as a purely predictive theory, without the recourse to measured data, SEA has not been universally successful. Nevertheless in some cases it has been very successful, for example when used to model the interaction between the noise in a room and its vibrating walls, but when applied to complex structural assemblies SEA predictions have often exhibited errors. These errors have been thought due to the fact that plates and beams are usually strongly coupled and one of the assumptions within standard SEA theory (see, for example, Lyon 1984) is that all couplings are weak. However, Keane & Price (1987) conclude that this assumption should be replaced by the necessity that no individual mode within a given sub-system should dominate the overall response of that sub-system, and this requirement can be met either by assuming weak coupling or by assuming the presence of many interacting modes. Furthermore, if SEA theory is developed using the wave approach rather than the modal approach this weak coupling assumption does not appear to be required (see, for example, Heron 1990).

In this paper we postulate that the errors that sometimes occur when predictive SEA is applied to complex structural assemblies are mainly due to an as yet unmodelled power transport mechanism. This 'tunnelling' mechanism conceptually occurs when direct coupling exists between two SEA sub-systems that are physically separated from each other by other SEA sub-systems. This mechanism of indirect coupling must not be confused with the power transport mechanism by which plate

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in-plane motion can couple physically separate bending motions; this latter phenomenon is fully modelled by existing SEA theory provided the in-plane sub-systems are included in the model.

A very simple form of tunnelling is associated with the non-resonant acoustic transmission through a plate and is already included in existing SEA theory (see, for example, Price & Crocker 1970; Leppington *et al.* 1987). However, this special case is mainly a function of the change in the dimension between the plate and the adjacent rooms and is not the concern of this paper.

Standard predictive SEA assumes zero coupling between the end plates of, for example, an in-plane assembly of three in-line plates. In this paper we develop a theory that allows for all sub-systems to be coupled to each other. Unlike for the simpler case of non-resonant acoustic transmission through a plate, we would intuitively expect this new theory to produce coupling loss factors that are not only a function of the power transmission coefficients across the various intervening line junctions, but are also dependent on the geometry and damping of the intervening plates.

In the following sections this theory is developed both for beam and plate networks and for want of a better name we will subsequently refer to this theory as advanced SEA or simply ASEA. Fundamentally it uses a ray theory approach to track the power flowing around the network and then uses standard SEA to take care of any residual power.

2. Free and fixed energy

Now all deterministic theories (modal, analytic, FEM, etc.) use field variables such as displacement and pressure and they must therefore include phase in the model, and the very essence of a high frequency model is the simplification associated with ignoring these phase effects. It is not just the need for computational efficiency that drives us to this assumption, but as Hodges & Woodhouse (1986) point out as we move to higher frequencies any deterministic approach becomes increasingly sensitive to the details of the physical structure under investigation, to such an extent that the results will be influenced by the deviations from the ideal design that inevitably occur in construction and such deviations are unknown. Thus all such deterministic approaches are rejected in this paper without further consideration; power accounting and the use of the sub-system energies as the field variables are the mainstays of SEA, and ASEA will be developed using the same philosophy.

The tunnelling phenomenon that we are attempting to model is associated with the transport of power, from sub-system 1 to sub-system 3 via an intervening sub-system 2 without at the same time inducing any increase in the 'free energy' of the intervening sub-system, and we must now consider what we mean by free energy.

With free energy we mean that part of the total sub-system energy that is available for transport to other sub-systems. In standard SEA all sub-system energy is free energy. Conversely the fixed energy of a sub-system is that part of the total sub-system energy that is not available for transport to other sub-systems. This postulates that the total energy of a SEA sub-system can be partitioned into a free and a fixed part is fundamental to ASEA theory.

Returning to the three in-line plate assembly example, we can now consider the following power flow mechanism. Free power associated with the free energy of sub-system 1 strikes the line junction between plate 1 and plate 2, this causes some power to transmit into plate 2 and as this power transports across plate 2 it will decrease

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in magnitude due to the damping mechanisms of plate 2. It is this loss of power that is self evidently not available for further transport duties and must be accounted for by a fixed energy field within plate 2. Finally some part of this transported power will strike the line junction between plates 2 and 3 where it will cause power injection into plate 3, and at this level of complexity such power will feed into the free energy field of plate 3.

3. SEA basics

First we find it helpful to rewrite the standard SEA matrix equation in a more convenient form for subsequent extension to ASEA such that

$$A\boldsymbol{e} = \boldsymbol{P} - M\boldsymbol{e}, \tag{3.1}$$

where $\boldsymbol{e}$ is a column vector of SEA modal energies, $\boldsymbol{P}$ is a column vector of input powers, M is a diagonal matrix of modal overlap factors, and A is a matrix of coupling loss factors. That is

$$M_{ii} = \omega n_i \eta_i, \tag{3.2}$$

where ω is the frequency, n_i is the modal density of sub-system i and η_i is the energy loss factor for sub-system i; furthermore for a three sub-system model we have

$$A = \omega \begin{bmatrix} n_1\eta_{12} + n_1\eta_{13} & -n_2\eta_{21} & -n_3\eta_{31} \\ -n_1\eta_{12} & n_2\eta_{21} + n_2\eta_{23} & -n_3\eta_{32} \\ -n_1\eta_{13} & -n_2\eta_{23} & n_3\eta_{31} + n_3\eta_{32} \end{bmatrix}, \tag{3.3}$$

where η_{ij} is the usual SEA coupling loss factor.

Of course the more usual SEA matrix equation can be recovered by combining the A and the M matrices in (3.1). The reason for the above formulation will become apparent as we develop ASEA theory, but for now it is worth noting that each of the three terms in (3.1) have a clear physical meaning; the left-hand side term incorporates all the power transport and coupling effects and the two right-hand side terms model all the power sources and all the power sinks respectively.

Furthermore, if all the equations in (3.1) are added together we have, by power balance, that the sum of all the right-hand side terms is zero, and this is true for all possible $\boldsymbol{P}$ and thus for all possible $\boldsymbol{e}$. Hence each individual column of A must always sum to zero, which is of course a trivial deduction from SEA. Indeed, assuming SEA reciprocity, A is a symmetric matrix and thus each individual row of A must also sum to zero. However, it is important to note that this row sum rule is a consequence of power balance and SEA reciprocity whereas the column sum rule is solely a consequence of the much more fundamental requirement of power balance.

4. ASEA basics

In developing ASEA theory we will, as described above, split the total energy field within each sub-system into two parts, a free energy field with a modal energy of e, and a fixed energy field with an 'equivalent' modal energy of d. The term 'modal energy' is used because of its historic link with classical SEA theory, however the reader might find it easier to think of the modal energy as a measure of the energy density of a sub-system with which it is closely related for sub-systems made up of simple beams, plates or rooms.

Using the column vectors $\boldsymbol{e}$ and $\boldsymbol{d}$ as the field variables, ASEA can be encapsulated by the following two matrix equations

$$\underset{\text{free power to free power transfer}}{A\boldsymbol{e}} = \underset{\text{free power input}}{\boldsymbol{P}} - \underset{\text{free power lost}}{M\boldsymbol{e}}, \tag{4.1}$$

$$\underset{\text{free power to fixed power transfer}}{B\boldsymbol{e}} = \underset{\text{fixed power input}}{\boldsymbol{Q}} - \underset{\text{fixed power lost}}{M\boldsymbol{d}}, \tag{4.2}$$

and to understand better these equations we have attached a physical description to each of the terms. The above equations form the basis for ASEA theory and this paper is mainly concerned with the calculation procedure for the A and the B matrices.

It may be thought that the somewhat arbitrary use of M in the second equation involves an assumption but this is not so since we have yet to specify the precise definition of B and $\boldsymbol{Q}$, and the requirement to conform with equation (4.2) creates those definitions.

Once A, B, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are known the responses can be calculated from $\boldsymbol{e}+\boldsymbol{d}$, using exactly the same procedures that we currently use when calculating SEA responses from $\boldsymbol{e}$. It should be noted that the A matrix of ASEA theory is not the same as the A matrix of standard SEA theory.

From equations (4.1) and (4.2), $\boldsymbol{e}+\boldsymbol{d}$ is given by

$$\boldsymbol{e}+\boldsymbol{d} = M^{-1}(\boldsymbol{Q}+\boldsymbol{R}), \tag{4.3}$$

where

$$\boldsymbol{R} = (M-B)(M+A)^{-1}\boldsymbol{P}. \tag{4.4}$$

Now for the classical excitation of 'rain on the roof' $\boldsymbol{Q}$ is zero, and with this simplification equation (4.3) can be rewritten as

$$(M+A)(M-B)^{-1}M(\boldsymbol{e}+\boldsymbol{d}) = \boldsymbol{P}, \tag{4.5}$$

and this equation can be considered to be the 'equivalent' standard SEA matrix equation such that if

$$A_{\text{sea}}\boldsymbol{e}_{\text{sea}} = \boldsymbol{P}, \tag{4.6}$$

then

$$A_{\text{sea}} = (M+A)(M-B)^{-1}M, \tag{4.7}$$

and

$$\boldsymbol{e}_{\text{sea}} = \boldsymbol{e}+\boldsymbol{d}. \tag{4.8}$$

Finally by applying the same power balancing argument of §3 we can easily deduce the important property that each individual column of $A+B$ must always sum to zero.

5. ASEA and beam networks

In a beam network each beam will consist of four sub-systems associated with its two bending wavetypes, its compressional wavetype and its torsional wavetype. In this section, for clarity of presentation, we will only consider a network of rods with each rod having only one wavetype. Provided we allow for this one wavetype to be conceptually of any type, for example by not assuming that the group velocity is equal to the phase velocity, then the extension to a beam network is straightforward.

Consider now the free energy field of rod j, represented by its modal energy e_j. Then the total free energy of this rod, E_j, is given by

$$E_j = n_j e_j, \tag{5.1}$$

and the energy density of this free energy is E_j divided by L_j, where L_j is the length of rod j. Now by assuming that this energy field is made up of equal amounts of incoherent power, P_j, flowing both from left to right and from right to left along the rod (equivalent to the random incidence assumption in two- and three-dimensional sub-systems) we have

$$E_j/L_j = 2P_j/c_{gj}, \tag{5.2}$$

where c_{gj} is the group velocity of rod j.

Furthermore since for all one-dimensional sub-systems

$$n_j = L_j/\pi c_{gj}, \tag{5.3}$$

we can combine equation (5.1) and equation (5.2) to obtain the standard SEA result that

$$P_j = e_j/2\pi. \tag{5.4}$$

Thus for unit modal energy the power available at each end of rod j, P_{aj} say, for potential transportation to the other rods, is simply $\frac{1}{2}\pi$.

We can now proceed with the calculation of the elements of the matrices A and B. Initially all these are set to zero and the calculation is based on using the elements of these matrices as accumulators. We start by taking a particular end of a particular rod and ultimately repeat the calculation for both ends of every rod.

The power available per unit modal energy P_{aj} at this particular end of rod j will conceptually be all transferred from rod j, and thus P_{aj} must now be added to element (j,j) of matrix A; add rather than subtract because the transfer terms have been conventionally placed on the left-hand side of equation (4.1) and equation (4.2).

Now we take this available power, P_{aj} say, and multiply it by the appropriate transmission or reflection coefficient. This is then the power at the connected end of a particular receiving rod, rod i say, and this power is now ready for transportation across this rod; rod i can be the same rod as rod j to take care of the reflected wave and indeed the following calculations must be performed for all rods connected to the chosen end of rod j including rod j itself. This start power, P_{si} say, is thus given by

$$P_{si} = \tau_{ij} P_{aj}, \tag{5.5}$$

where τ_{ij} is the power transmission coefficient for power flowing from rod j to rod i.

It is worth keeping in mind at this point the standard SEA theory which would proceed in the following manner

$$\omega n_j \eta_{ji} = P_{si} = \tau_{ij} P_{aj} = \tau_{ij}/2\pi, \tag{5.6}$$

and thus
$$\eta_{ji} = \tau_{ij}/2\pi\omega n_j. \tag{5.7}$$

Returning to ASEA theory, power will flow across rod i and will decay as it does so with the exponential factor

$$\exp(-\omega\eta_i L_i/c_{gi}) = \exp(-\pi M_i), \tag{5.8}$$

where M_i is the modal overlap factor of rod i. Thus

$$P_{ei} = \exp(-\pi M_i) P_{si}, \tag{5.9}$$

where P_{ei} is the power striking the far end of rod i. The power lost during this crossing, P_{li} say, is given by

$$P_{li} = P_{si} - P_{ei}. \tag{5.10}$$

This lost power must now be subtracted from element (i,j) of matrix B; matrix B rather than matrix A since this power is self evidently unavailable for further transport duties. On the other hand, P_{ei} is available for further transport duties, and indeed we can continue the calculation from equation (5.5) using P_{ei} rather than P_{aj}. Of course within this cycle of the calculation we can only modify column j of either matrix A or matrix B since all of the initial available power comes from rod j. This whole process can be stopped at any stage and having stopped any remaining power, P_{si} say, must then be subtracted from element (i,j) of matrix A. This latter is essential to maintain power balance and conceptually uses a standard SEA approach to sweep up and account for the residual power P_{si}; it also ensures that all the columns of $A+B$ sum to zero as required by power balance.

6. ASEA and plate networks

The above theory can be extended to plate networks although its actual implementation could well turn out to be computationally expensive, as compared with standard SEA. However, ASEA plate theory will hopefully guarantee an accurate prediction and the fact that it may not become a practical tool because of the computational load should not deter us from its development. Its use as a tool for the validation of more approximate theories is very important because no accurate high frequency theory exists for general structural assemblies.

Whereas with rods we calculated the A and the B matrices by starting with a particular end of a particular rod and with beams we would start with a particular end of a particular beam and with a particular wavetype, with plates we must start with a particular edge of a particular plate and not only with a particular wavetype but also with a particular incidence angle at the chosen edge. In standard SEA the eventual integral over all angles of incidence is carried out implicitly within the model such that the formula for an SEA plate to plate coupling loss factor is a function of the random incidence transmission coefficient as given below in equation (6.6). In ASEA we can only perform the integral over all possible angles of incidence, 180°, at the end of the A and B calculation; although by converting this integral into a suitably weighted sum we can easily incorporate it into the calculation procedure. Unfortunately line junction transmission coefficients tend to vary a lot with angle of incidence due mainly to the complex interaction effects of the various wavetypes and it is often necessary to perform these calculations over many angles of incidence: typically at every integer degree.

For a random diffuse energy field in sub-system j of a plate the intensity, I_j say, is given by

$$I_j = e_j k_j / 4\pi, \tag{6.1}$$

where k_j and e_j are the wavenumber and modal energy respectively of the wavetype associated with sub-system j. The power per unit modal energy striking an edge of length L at a grazing angle of incidence ϕ_j is thus

$$P_{aj} = Lk_j \sin(\phi_j)/4\pi, \tag{6.2}$$

and as before this must now be added to element (j,j) of matrix A.

We set

$$P_{si} = \tau_{ij}(\phi_j)\, P_{aj}, \tag{6.3}$$

however, τ_{ij} is now a function of ϕ_j and the transmitted wave angle has to be calculated using trace wavenumber matching such that

$$k_i \cos(\phi_i) = k_j \cos(\phi_j). \tag{6.4}$$

Again at this point it is worth keeping in mind the standard SEA theory which for plates proceeds as follows

$$\omega n_j \eta_{ji} = \pi^{-1} \int_0^{\pi} P_{si}\, d\phi_j, \tag{6.5}$$

and thus

$$\eta_{ji} = L k_j \hat{\tau}_{ij} / 2\pi^2 \omega n_j, \tag{6.6}$$

where the random incidence transmission coefficient is given by

$$\hat{\tau}_{ij} = \int_0^{\pi} \tau_{ij}(\phi_j) \sin(\phi_j)\, d\phi_j. \tag{6.7}$$

Returning to ASEA theory, geometric calculations must now be made to track the wave as it is transported across sub-system i. This can result in more than one edge of the plate supporting sub-system i being illuminated and furthermore an illuminated edge need not be illuminated along its entire length; both of these effects must be calculated.

The damping factor, equivalent to the factor $e^{-\pi M}$ of equation (5.8), is also more complicated here. Different parts of the wave will travel different distances, however for polygon shaped plates a damping factor averaged over all possible path lengths between two edges can be used and this is given by

$$\frac{(e^{-b\kappa} - e^{-a\kappa})}{(a\kappa - b\kappa)} = D, \tag{6.8}$$

where

$$\kappa = \omega \eta_i / c_{gi} = 2\pi M_i / A_i k_i, \tag{6.9}$$

and where a and b are the maximum and minimum path lengths.

Finally

$$P_{ei} = D P_{si} \tag{6.10}$$

and

$$P_{1i} = P_{si} - P_{ei}, \tag{6.11}$$

and P_{1i} is subtracted from element (i,j) of matrix B as before.

7. Comparison with analytical results

ASEA produces a different result dependent on the number of transfers of power across a sub-system that we are modelling. This number which is also one less than the number of junctions crossed we will call the ASEA level number, and with a level number of zero ASEA always produces results identical to standard SEA since both B and $\boldsymbol{d}$ are then zero. Advanced SEA can thus be thought of as a series of approximations,

$$\text{ASEA}_0\ (\equiv \text{SEA}),\ \text{ASEA}_1,\ \text{ASEA}_2,\ \text{ASEA}_3,\ \ldots, \tag{7.1}$$

with the expectation that this series converges on the required result.

It is important to understand why we have this clear expectation that if the series (7.1) converges at all it must converge onto the 'correct' result; correct in the sense of giving the best high frequency result possible.

Consider the calculation procedure for ASEA with a very large level number; the level number chosen to be so large as to cause the A matrix to be effectively zero. Then the ASEA calculation procedure is nothing more than ray tracing with all phase related effects ignored, or in other words simple power flow analysis. But unless we want to encroach on the low frequency deterministic domain, any high frequency theory must at least make the assumption, explicitly or implicitly, that all phase effects be ignored. Now with this assumption, and this assumption alone, we can deduce ASEA for an infinite level number. (Self evidently this would also be true for a simple power flow analysis, the subtle difference is that ASEA hopefully converges much faster due to the different treatment of the 'remainder' terms, which are ignored in a simple power flow analysis but are injected into a SEA procedure whose results are added to the truncated power flow analysis during an ASEA calculation.) Thus we fully expect that, if ASEA converges at all, and if an accurate high frequency theory exists at all, ASEA will converge onto the best theoretical result possible.

To show this convergence for a particular case we have chosen a very simple assembly consisting of six different rods all in a line. In principle an assembly of plates could equally well have been chosen; however, exact results are extremely difficult to compute for plate assemblies at high frequencies and thus we have chosen an assembly of rods. The inline configuration has been deliberately chosen to highlight the errors in a simple SEA calculation and the subsequent correction of these errors by ASEA. The inability of SEA to predict such a contrived configuration is understandable and does not detract from the usefulness of SEA when applied to more realistic structures, but it should be considered as a warning that the accuracy of SEA is structure dependent.

The coupling between the rods is such that conceptually the whole structure could be made from a single piece of material with the far ends of the chain left unsupported or free. The rod material is such that its longitudinal phase, or group, velocity is 5000 m s^{-1}. The six rods are of lengths 23, 28, 25, 24, 29 and 21 m and their cross-sectional areas are such that their mass per unit lengths are 1, 10, 3, 7, 8 and 2 kg m^{-1} respectively. An energy damping value of 2% was chosen for the SEA modelling, and viscous damping with an equivalent critical damping ratio of 0.01 chosen for the exact model. The structure was always driven with a unit force on the first rod.

The exact results were calculated by Keane (1992; personal communication) and form a full deterministic analysis for point excitation, they are based on calculating the power flow across the assumed point connections between the rods for a given unit point force excitation on the drive rod. These response data were then numerically averaged over all excitation positions on the drive rod, rod 1, and over all frequencies within the chosen frequency bandwidth of 50 Hz.

Figure 1*a–d* shows the results for the averaged response on the four rods furthest from the drive rod; the results for rods 1 and 2 are not shown because SEA and all levels of ASEA lie very close to the exact results for these rods. All the displayed responses have been normalized to unit mean square response velocity at the drive point on rod 1.

As can be seen SEA, or equivalently ASEA_0, is not an adequate model at the higher frequencies; at 10 kHz SEA over predicts the response of rod 6 by over 60 dB. On the other hand, as expected, ASEA always predicts accurately provided we are willing to calculate to a high enough level number. For a chain of rods driven at one end the rule of convergence appears to be that the ASEA level number should be at least the rod number minus two. This is not so surprising a result since such a level number ensures a direct coupling exist in the ASEA model between the drive rod and the response rod. The convergence of ASEA is not necessarily monotonic with level number as can be seen in figure 1*d*, where ASEA_2 gives a slightly better result than ASEA_3.

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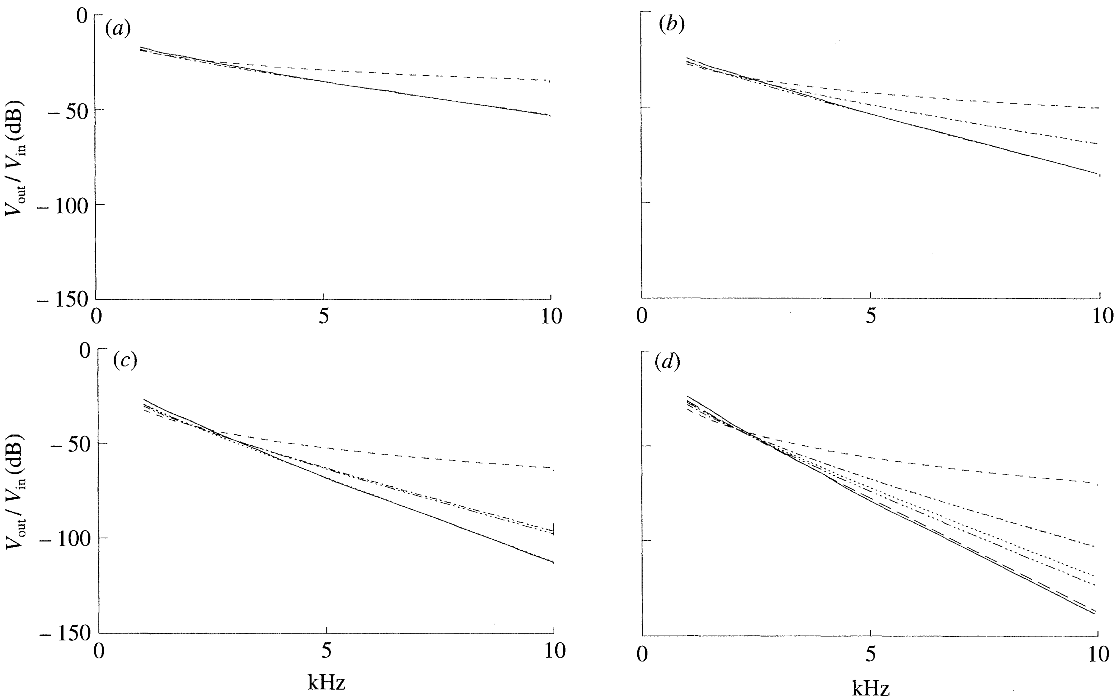

Figure 1. Response of rod 3 (*a*), 4 (*b*), 5 (*c*) and 6 (*d*). ——, Exact result; ----, ASEA$_0$ prediction; —·—, ASEA$_1$ prediction; —···—, ASEA$_2$ prediction; ....., ASEA$_3$ prediction; — —, ASEA$_4$ prediction.

8. Conclusions

A high frequency theory (ASEA) has been presented that takes account of the mechanism of tunnelling. This mechanism which requires the introduction of coupling between SEA sub-systems that are physically separate is modelled by creating a new set of basic ASEA equations and dividing the energy of a sub-system into energy that is freely available for transfer to other sub-systems and energy that is fixed within the sub-system.

These equations are presented and an attempt has been made to give their component parts physical meaning. The calculation procedure is presented for modelling either a general beam network or a general plate network. ASEA is interpreted as a series of mathematical models, the first of which is identical to standard SEA and subsequent higher order models are convergent on the desired result.

Using a structural assembly of six rods as an example, ASEA converges onto the exact results whereas SEA is shown to overpredict by up to 60 kB.

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